PLEASE ANSWER ALL QUESTIONS.
PLEASE EXPLAIN YOUR ANSWERS.

1. (a) Find all the pure and mixed-strategy Nash Equilibria of the following game.

Player 1 \ Player 2	t_1	t_2	t_3
s_1	$1, 0$	$5, 2$	$1, 5$
s_2	$3, 3$	$2, 1$	$0, 2$

Solution: There are two pure-strategy NE: (s_2, t_1) and (s_1, t_3). For the mixed-strategy equilibrium, let P1's strategy be denoted $(p, 1-p)$ and P2's be denoted $(q_1, q_2, 1-q_1-q_2)$. Notice that t_2 is strictly dominated by t_3, so in equilibrium $q_2 = 0$.
Thus, the players are indifferent between their (non-dominated strategies) when

$$q_1(1) + (1-q_1)(1) = q_1(3) + (1-q_1)(0) \Leftrightarrow q_1 = 1/3$$
$$p(0) + (1-p)(3) = p(5) + (1-p)(2) \Leftrightarrow p = 1/6.$$

So the mixed-strategy NE is $(p; q_1, q_2) = (1/6; 1/3, 0)$.

(b) Suppose now that we introduce a new strategy for Player 1. Denote the corresponding game by G:

Player 1 \ Player 2	t_1	t_2	t_3
s_1	$1, 0$	$3, 2$	$1, 5$
s_2	$3, 3$	$2, 1$	$0, 2$
s_3	$0, 4$	$10, 10$	$0, 11$

Use iterated elimination of strictly dominated strategies to simplify the game. Explain briefly each step (1 sentence). What is the set of pure and mixed-strategy Nash Equilibria of G?

Solution: Again, t_2 is strictly dominated by t_3. After eliminating t_2, then s_3 is strictly dominated by s_1. After eliminating s_3, no strategies are strictly dominated. This game is equal to the game in (a) after eliminating the strictly dominated strategy t_2. Hence, the set of NE is the same in the two games.

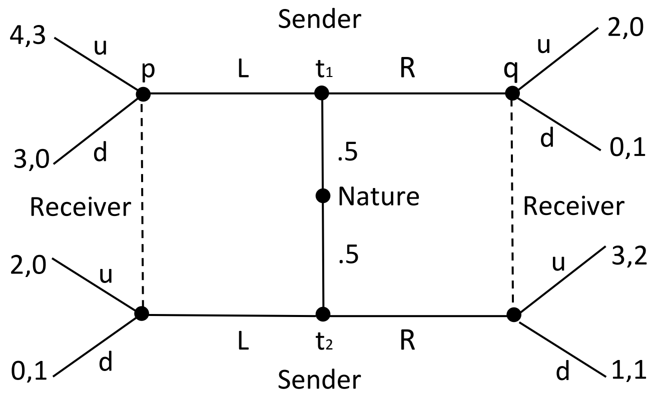
(c) Now suppose we repeat G twice. Denote the resulting game by $G(2)$. How many proper subgames are there (not counting the game itself)? Show that there is a Subgame-perfect Nash Equilibrium of $G(2)$ in which (s_3, t_2) is played in stage 1.

Solution: One proper subgame after each possible outcome in G: 9 proper subgames.

Proposed equilibrium strategies: in stage 1, play (s_3, t_2); in stage 2, play (s_1, t_3) on the equilibrium path and (s_2, t_1) off the equilibrium path.

Check deviations: In stage 2, a NE is played in each subgame, so no profitable deviations. In stage 1, P1 gets $10 + 1 = 11$ on the equilibrium path, and at most $3 + 3 < 11$ from a deviation. P2 gets $10 + 5 = 15$ on the equilibrium path, and at most $11 + 3 < 15$ from a deviation. Hence, the proposed equilibrium strategies form a SPNE.

2. **Signaling.** Consider the following signaling game.



(a) Find all the (pure strategy) separating Perfect Bayesian Equilibria (PBE).

Solution: $(LR, uu; p = 1, q = 0)$ is the unique separating PBE.

Case 1. Suppose $m(t_1) = L$ and $m(t_2) = R$. Then $p = 1$ and $q = 0$. Thus, $a(L) = u$ and $a(R) = u$. Can check that $u_S(L, u; t_1) \geq u_S(R, u; t_1)$ and $u_S(R, u; t_2) \geq u_S(L, u; t_2)$ hold. Hence: PBE.

Case 2. Suppose $m(t_1) = R$ and $m(t_2) = L$. Then $p = 0$ and $q = 1$. Thus, $a(L) = d$ and $a(R) = d$. Can verify that $u_S(R, d; t_1) < u_S(L, d; t_1)$. Hence, not a PBE.

(b) Find the (pure strategy) pooling equilibrium in which both types send message L. Does it satisfy signaling requirement 5 (SR5)?

Solution: Suppose $m(t_1) = m(t_2) = L$. Then $a(L) = u$ (since $\frac{1}{2}(3) + \frac{1}{2}(0) > \frac{1}{2}(0) + \frac{1}{2}(1)$). Check sender's incentives: $u_S(L, u; t_1) \geq u_S(R, a(R); t_1)$ for all $a(R)$ whereas $u_S(L, u; t_2) \geq u_S(R, a(R); t_2)$ only if $a(R) = d$. It is optimal for the receiver to choose $a(R) = d$ if

$$q(1) + (1 - q)(1) \geq q(0) + (1 - q)(2) \Leftrightarrow q \geq 1/2.$$

Thus: $(LL, ud; p = 1/2, q \geq 1/2)$ is a pooling PBE.

Notice that R is strictly dominated by L for t_1, but not for t_2. Therefore, SR5 prescribes that $q = 0$. Hence, the pooling PBE we just found does not satisfy SR5.

(c) Explain in your own words the logic behind SR5. You may use the above game as an example.

Solution: SR5 is based on the idea of forward induction, and attempts to capture the intuition that no players should play strictly dominated strategies. Thus, in the above example, since playing R is strictly dominated for type 1 but not for type 2, it seems more reasonable to think that a potential deviator is type 2.

3. Consider a *second-price sealed bid auction* with two bidders, who have valuations v_1 and v_2, respectively.

(a) First, assume that the values are distributed independently uniformly with

$$v_i \sim u(1, 2).$$

Thus, the values are **private**. Show that there is a symmetric Bayesian Nash Equilibrium where the players bid their valuation: $b_i(v_i) = v_i$ (recall that the auction format is second-price sealed bid).

(Hint: Look at whether the players can profitably deviate by bidding higher or lower.)

Solution: Throughout suppose that j sticks to his equilibrium strategy: $b_j = v_j$. The probability that two bids are the same is zero, and therefore we only consider 'inequalities'.

Suppose player i deviates by bidding $b' < v_i$. If $v_j > v_i$ then $b' < b_j$ and player i loses in either case. If $v_j < b' < v_i$ then player i wins and pays $p = v_j$ in either case. If $b' < v_j < v_i$ then player i wins and gets payoff $v_i - v_j > 0$ if he sticks to the equilibrium strategy, and he loses and gets payoff 0 if he deviates. Thus, $b' < v_i$ is never a profitable deviation.

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Hence, bidding b_j is weakly optimal for both players, and therefore a NE.

(b) Consider now the following **common value** setting. The auction format is still *second price*. Each player i observes a signal s_i, where

$$s_i \sim u(1,2).$$

The valuation of the players is the sum of the two signals: for each i,

$$v_i = s_1 + s_2.$$

The expected valuation of player i conditional on s_i is $\mathbb{E}[v_i|s_i] = \mathbb{E}[s_1+s_2|s_i] = s_i+\frac{3}{2}$. Suppose players bid their expectation, i.e. that $b_i(s_i) = s_i + \frac{3}{2}$. What is the expected value of player i conditional on s_i and conditional on winning the auction? I.e., what is $\mathbb{E}[v_i|s_i, i \text{ wins}]$.

Solution: The expectation is

$$\begin{aligned}
\mathbb{E}[v_i|s_i, i \text{ wins}] &= \mathbb{E}[v_i|s_i, s_i \geq s_j] \\
&= \mathbb{E}[s_i + s_j|s_i, s_i \geq s_j] \\
&= s_i + \mathbb{E}[s_j|s_i, s_i \geq s_j] \\
&= s_i + \frac{1+s_i}{2} \\
&< s_i + \frac{3}{2}.
\end{aligned}$$

(c) Relate your answer in the last question to the concept of the *winner's curse*.

Solution: For player i, winning the auction means (in equilibrium) that the signal of player j was lower than i's signal. Thus, winning the auction is 'bad news' for player i, in the sense that it lowers his valuation.

4. Consider the following exercise in which a buyer and a seller have valuations v_b and v_s, but only the seller knows the valuations. The buyer makes an offer of a price, and the seller chooses whether to accept. The details are as follows.

Valuations. The seller's valuation is uniformly distributed on the unit interval. I.e.

$$v_s \sim u(0,1).$$

The buyer's valuation is $v_b = k \cdot v_s$, where $k > 1$ is common knowledge.

Information. Seller knows v_s (and hence v_b) but the buyer does **not** know v_b (or v_s).

Buyer. The buyer makes a single offer, p, which the seller either accepts ($a = 1$) or rejects ($a = 0$). (I.e., it is the *buyer* who sets the price, and seller who decides whether he accepts or rejects.) The buyer gets payoffs

$$u_b(p, a) = \begin{cases} v_b - p \text{ if } a = 1 \text{ (seller accepts)}, \\ 0 \text{ if } a = 0 \text{ (seller rejects)}. \end{cases}$$

The buyer's strategy is just a choice of p, since he cannot condition his choice on v_b.

Seller. The seller's payoffs are

$$u_s(p, a) = \begin{cases} p \text{ if } a = 1 \text{ (seller accepts)}, \\ v_s \text{ if } a = 0 \text{ (seller rejects)}. \end{cases}$$

His strategy can be described as a function $a(p, v_s)$, where $a(p, v_s) = 1$ corresponds to accepting the offer of p when his valuation is v_s, and $a(p, v_s) = 0$ corresponds to rejecting it. Suppose that whenever he is indifferent, he accepts the offer.

We will look for a Perfect Bayesian Equilibrium (PBE).

(a) Show that in a PBE, $a^*(p, v_s) = 1$ if and only if $v_s \leq p$.
Solution: PBE requires the players to maximize utility in each information set, given their beliefs. Seller perfectly knows his valuation, so therefore his payoff from selling is p and his payoff from not selling is v_s. Thus, he sells only if $v_s \leq p$.

(b) Buyer's expected payoff from making an offer of p is

$$\pi(p)(\mathbb{E}[v_b|\text{seller accepts}, p] - p),$$

where $\pi(p) = \mathbb{P}(\text{seller accepts}|p)$.

i. Find $\pi(p)$ given $a^*(p, v_s)$.
ii. Find $\mathbb{E}[v_b|\text{seller accepts}, p]$ given $a^*(p, v_s)$.

Solution: Using standard results on uniform distributions, then given a^* we have $\pi(p) = \mathbb{P}(v_s \leq p) = p$ for $p \in [0, 1]$ and $\pi(p) = 1$ for $p > 1$. Furthermore,

$$\mathbb{E}[v_b|\text{seller accepts}, p] = k\mathbb{E}[v_s|v_s \leq p, p] = k \cdot \frac{p}{2},$$

for $p \in [0, 1]$ and $k \cdot \frac{1}{2}$ for $p > 1$.

(c) What is the PBE when $k > 2$? What is the probability that trade takes place? How would the answer change if $k < 2$?
Solution: Buyer's expected payoff for $p \in [0, 1]$ are

$$p \cdot (k \cdot \frac{p}{2} - p) = p^2 \cdot (\frac{k}{2} - 1). \tag{1}$$

For $p > 1$ the payoffs are $\frac{k}{2} - p$. Clearly, for $k > 2$, payoffs are strictly increasing for $p \in [0, 1)$ and strictly decreasing for $p > 1$. Continuity implies that the expected payoffs are maximized at $p^* = 1$. The PBE is $(p^* = 1, a^*(p, v_s))$, where $a^*(p, v_s)$ is as above. Trade always takes place.

For $k < 2$, payoffs are strictly decreasing for $p > 0$. Therefore, expected payoffs are maximized at $p^* = 0$. The PBE is $(p^* = 0, a^*(p, v_s))$, where $a^*(p, v_s)$ is as above. Trade never takes place. The truly excellent answer might note that there is a type of winner's curse at play here.